Assignment 2

1. Build a Hodgkin-Huxley model neuron by numerically integrating the equations for $V$, $m$, $h$, and $n$:

$$c_m \frac{dV}{dt} = -i_m + \frac{I_e}{A},$$

where

$$i_m = g_L(V - E_L) + g_K n^4(V - E_K) + g_{Na} m^3 h(V - E_{Na}).$$

and

$$\tau_n(V) \frac{dn}{dt} = n_{\infty}(V) - n,$$

$$\tau_n(V) = \frac{1}{\alpha_n(V) + \beta_n(V)}$$

and

$$n_{\infty}(V) = \frac{\alpha_n(V)}{\alpha_n(V) + \beta_n(V)},$$

and similar equations for $m$ and $h$, with

$$\alpha_n = \frac{.01(V + 55)}{1 - \exp(-.1(V + 55))} \quad \beta_n = 0.125 \exp(-0.0125(V + 65)),$$

$$\alpha_m = \frac{.1(V + 40)}{1 - \exp(-.1(V + 40))} \quad \beta_m = 4 \exp(-.0556(V + 65)),$$

$$\alpha_h = .07 \exp(-.05(V + 65)) \quad \beta_h = 1/(1 + \exp(-.1(V + 35))).$$

In these equations, time is in ms and voltage is in mV. Take $c_m = 10$ nF/mm$^2$ and, as initial values, take: $V = -65$ mV, $m = 0.0529$, $h = 0.5961$, and $n = 0.3177$. The maximal conductances and reversal potentials used in the model are $g_L = 0.003$ mS/mm$^2$, $g_K = 0.36$ mS/mm$^2$, $g_{Na} = 1.2$ mS/mm$^2$, $E_L = -54.387$ mV, $E_K = -77$ mV and $E_{Na} = 50$ mV. The best method to use to integrate all of these equations over time is the exponential integration scheme discussed in class, but any routine that works is OK. Make your integration step size small enough to get accurate results.

a) Use an external current with $I_e/A = 200$ nA/mm$^2$ and plot $V$, $m$, $h$, and $n$ as functions of time for a suitable interval.

b) Plot the firing rate of the model as a function of $I_e/A$ over the range from 0 to 500 nA/mm$^2$.

c) Apply a pulse of negative current with $I_e/A = -50$ nA/mm$^2$ for 5 ms followed by $I_e/A = 0$ and show what happens. Why does this occur?
2. Simulate the pair of equations

\[
\frac{dx}{dt} = -x(x-1)(x+1) + y \quad \frac{dy}{dt} = -k(2x + y)
\]

with different values of \( k \) in the range \( 0 < k < 2 \).

a) Find the largest value of \( k \) for which these equations produce oscillations.

b) Compute this value analytically by considering the fixed point \( x = y = 0 \) and finding the value of \( k \) where this state makes the transition from being stable to being unstable. When this state is stable, is the approach to the fixed point exponential or oscillatory? When the state is unstable, is the motion away from the fixed point exponential or oscillatory?

c) Show that your analytic results on the approach to and escape from the fixed point match your simulation results.

3. **Optional Problem:** Compare spike shapes and firing-rate versus input current current curves for the linear, quadratic, and exponential integrate-and-fire models. The equations of all three models have the form

\[
\tau_m \frac{dV}{dt} = E - V + F(V) + R_m I_e.
\]

With \( F(V) = 0 \) for the linear integrate-and-fire model,

\[
F(V) = \frac{(V - E)^2}{\Delta_Q}
\]

for the quadratic integrate-and-fire model, and

\[
F(V) = \Delta_E \exp\left(\frac{V - V_T}{\Delta_E}\right)
\]

for the exponential integrate-and-fire model. Use the parameters \( E = V_{\text{reset}} = -65 \) mV, \( V_{\text{th}} = -50 \) mV, \( \tau_m = 10 \) ms, and \( R_m = 10 \) M\( \Omega \), and \( \Delta_Q = 2.5 \) mV, \( \Delta_E = 5 \) mV, \( V_T = -55 \) mV, and \( V_{\text{max}} = 30 \) mV.

For the linear integrate-and-fire model, reset the potential to \( V = V_{\text{reset}} \) whenever it gets to or above \( V_{\text{th}} \). For the other two models, reset the potential to \( V = V_{\text{reset}} \) whenever it gets to or above \( V_{\text{max}} \).

You can integrate these equations using the Euler method or, for the integrate-and-fire model only, the exponential method discussed in class, or use one of the Matlab integration routines.

4. **Optional Problem:** Show that the equation of the quadratic integrate-and-fire model,

\[
\tau_m \frac{dV}{dt} = E - V + \frac{(V - E)^2}{\Delta_y} + R_m I_e,
\]
can be turned into the equation

\[ \tau_m \frac{d\theta}{dt} = 1 - \cos(\theta) + (1 + \cos(\theta)) \left( \frac{R_m I_e}{\Delta V} - \frac{1}{4} \right), \]

by making the change of variables

\[ V(t) = E_L + \Delta V \left( \tan \left( \frac{\theta(t)}{2} \right) + \frac{1}{2} \right). \]

You will need some trigonometric identities.