\[ I_{\eta} = \sqrt{20} \tau \eta \quad \eta = \text{volts}^2/s \]

\[ \eta \sim \mathcal{N}(0, \sigma^2) \]

\[ x = \sqrt{20} \tau \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \eta(t') \]

\[ \langle x \rangle = \langle \eta \rangle = 0 \]

\[ \langle x^2(t) \rangle = 20 \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \int_{-\infty}^{t} dt'' e^{-(t''-t')/\tau} \]

\[ \langle \eta(t) \eta(t') \rangle = 20 \sigma^2 \delta_{\tau}(t) \int_{-\infty}^{t} dt'' e^{-(t''-t')/\tau} \]

\[ = 10 \gamma \delta_{\tau}^2 \delta_{\tau}(t) \]
\[ \frac{\Omega}{n} = \frac{1}{\Delta t} \]

\[ \langle \xi(t') \xi(t'') \rangle = \delta(t'-t'') \]

\[ \frac{\sigma^2}{\sigma_x^2} = N \sigma \quad D = \frac{\sigma_x^2}{\nu} \]

Suppose that, on a given trial, the potential moves from \( V \) to \( V + \Delta V \) in time \( \Delta t \). On average, the time it takes to get to the threshold from \( V + \Delta V \) must be \( \Delta t \) less than the time it takes from \( V \), so

\[ \langle T(V + \Delta V) \rangle = T(V) - \Delta t . \tag{1.34} \]

Expanding in a Taylor series,

\[ \langle T(V + \Delta V) \rangle \approx T(V) + T'(V)\langle \Delta V \rangle + \frac{1}{2} T''(V)\langle \Delta V^2 \rangle , \tag{1.35} \]

where the primes denote derivatives with respect to \( V \). Using equation 1.33,

\[ \langle \Delta V \rangle = \frac{(V_{ss} - V) \Delta t}{\tau_m} \quad \text{and} \quad \langle \Delta V^2 \rangle = 2D \Delta t , \tag{1.36} \]

we find, from 1.34, that

\[ DT''(V) + \frac{V_{ss} - V}{\tau_m} T'(V) + 1 = 0 . \tag{1.37} \]

Defining

\[ f(V) = \int^V dx \frac{(V_{ss} - V)}{\tau_m D} = -\frac{(V_{ss} - V)^2}{2\tau_m D} \tag{1.38} \]

so that \( f'(V) = (V_{ss} - V)/\tau_m D \), we can write down the solution to this equation as

\[ T'(V) = -\frac{e^{-f(V)}}{D} \int^V dy e^{f(y)} . \tag{1.39} \]

Integrating this result, we find

\[ T(V) = -\frac{1}{D} \int_{V_{th}}^V dx \ e^{-f(x)} \int_x^\infty dy \ e^{f(y)} , \tag{1.40} \]
where we have imposed the firing condition $T(V_{th}) = 0$. This means that the answer we seek is

$$\frac{1}{R} = \frac{1}{D} \int_{V_{reset}}^{V_{th}} dx \ e^{-f(x)} \int_{-\infty}^{x} dy \ e^{f(y)}. \quad (1.41)$$

With some substitution, this can be written as

$$\frac{1}{R} = \frac{\tau_m}{\sigma_V^2} \int_{V_{reset}}^{V_{th}} dx \ \exp((V_{ss} - x)^2/2\sigma_V^2) \int_{-\infty}^{x} dy \ \exp(-(V_{ss} - y)^2/2\sigma_V^2). \quad (1.42)$$

Finally, changing variables $y \rightarrow (y - V_{ss})/\sqrt{2}\sigma_V$ and $x \rightarrow (x - V_{ss})/\sqrt{2}\sigma_V$, we find

$$\frac{1}{R} = 2\tau_m \int_{(V_{reset} - V_{ss})/\sqrt{2}\sigma_V}^{(V_{th} - V_{ss})/\sqrt{2}\sigma_V} dx \ \exp(x^2) \int_{-\infty}^{x} dy \ \exp(-y^2). \quad (1.43)$$

Using the fact that

$$\int_{-\infty}^{x} dy \ \exp(-y^2) = \frac{\sqrt{\pi} (1 + \text{erf}(x))}{2}, \quad (1.44)$$

we obtain the final result

$$\frac{1}{R} = \tau_m \sqrt{\pi} \int_{(V_{reset} - V_{ss})/\sqrt{2}\sigma_V}^{(V_{th} - V_{ss})/\sqrt{2}\sigma_V} dx \ \exp(x^2) (1 + \text{erf}(x)). \quad (1.45)$$

**Useful Numerical Approximation**

The integral in equation 1.45 is difficult to compute numerically because of the nature of the integrand $\exp(x^2)(1 + \text{erf}(x))$. To compute this integral using standard methods, use the following approximation.

$$\exp(x^2)(1 + \text{erf}(x)) \approx \left\{ \begin{array}{ll}
\frac{f_1}{2} & \text{if } x \leq 0 \\
2 \exp(x^2) - f_1 & \text{if } x > 0,
\end{array} \right. \quad (1.46)$$

where

$$f_1 = t \exp(\alpha), \quad t = \frac{1}{1 + 0.5|x|}, \quad (1.47)$$

and

$$\alpha = a_1 + t(a_2 + t(a_3 + t(a_4 + t(a_5 + t(a_6 + t(a_7 + t(a_8 + t(a_9 + t(a_{10})))))))) \quad (1.48)$$

with

$$a_1 = -1.26551223 \quad a_2 = 1.00002368 \quad a_3 = 0.37409196 \quad (1.49)$$

$$a_4 = 0.09678418 \quad a_5 = -0.18628806 \quad a_6 = 0.27886087$$

$$a_7 = -1.13520398 \quad a_8 = 1.48851587 \quad a_9 = -0.82215223$$

$$a_{10} = 0.17087277$$