Random recurrent neural network (Sompolinsky, Crisanti, Sommers 1988):


N neurons

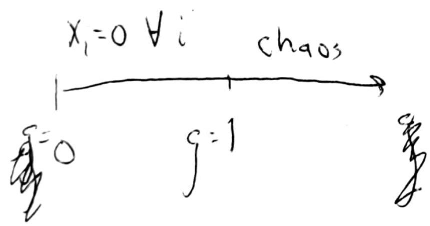
$$\tau \frac{dx_i}{dt} = -x_i + \sum_j J_{ij}\,\phi(x_j)$$

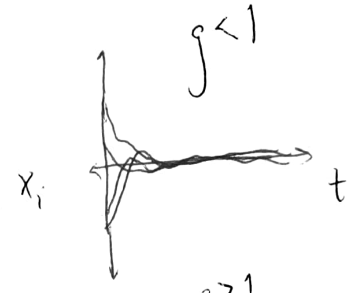
$$J_{ij} \sim N(0, g^2/N) \qquad \phi = \tanh$$


$g$: coupling strength

Why random?

1) Minimal model of disordered connectivity + nonlinearity
2) Initial condition for learning: $J = J_0 + M$, $M$ e.g. low-rank or symmetric
3) Rich internal dynamics (Arieli, Sterkin, Grinvald, Aertsen)



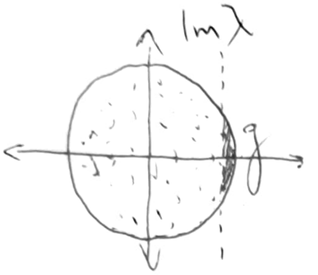


Linear stability at $\underline{x} = 0$:

$$\frac{dx_i}{dt} = -x_i + \sum_j J_{ij} x_j + O(x_j^2)$$

$$d\underline{x}/dt \approx (-\underline{I} + \underline{J})\,\underline{x}$$

Girko's circular law: If $J$ is $N \times N$ matrix w/ i.i.d. entries, ~~its eigenvalues~~ w/ mean $0$ and variance $g^2/N$, its eigenvalues distribution converges to unit disc of radius $g$ as $N \to \infty$

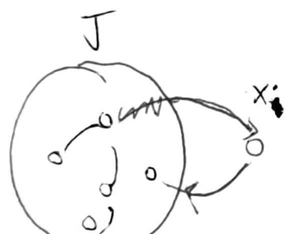



$g < 1$: Stable

$g > 1$: Unstable modes

Autocovariance $C_x(\tau) \equiv \langle x_i(t)\, x_i(t+\tau) \rangle_{t,i}$

$C_\phi(\tau) \equiv \langle \phi_i(t)\, \phi_i(t+\tau) \rangle_{t,i} \qquad \phi_i(t) = \phi(x_i(t))$



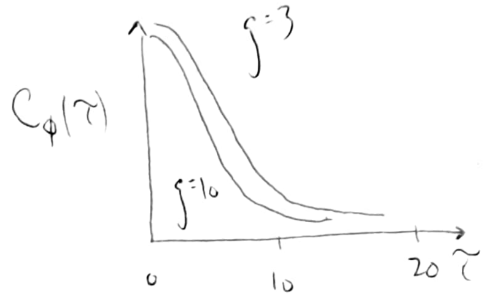

$$\boxed{\frac{dx_i}{dt} = -x_i + \eta_i(t)}$$

$$\langle \eta_i(t)\, \eta_i(t+\tau) \rangle_{t,\eta} = \Big\langle \sum_{j,k} J_{ij} J_{ik}\, \phi_j(t)\, \phi_k(t+\tau) \Big\rangle_{J,t}$$

$$\boxed{= g^2\, C_\phi(\tau)}$$

Solve for $C_\phi$ iteratively, or use

$$\frac{d^2}{d\tau^2} C_x(\tau) = -\frac{\partial}{\partial C_x} V(C_x)$$

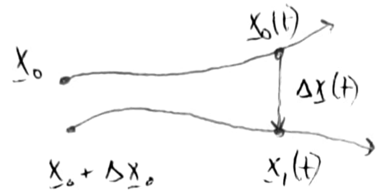



As $g - 1 \to 0$, decay time constant $\to 1/(g-1)$

As $g \to \infty$ $\qquad \to \sqrt{1 - 2/\pi}$

Chaos and Lyapunov exponent:

Consider 2 trajectories $\underline{x}_0(t)$ and $\underline{x}_1(t)$ with $\underline{x}_0(0) = \underline{x}_0$, $\underline{x}_1(0) = \underline{x}_0 + \Delta \underline{x}_0$. Let $\Delta \underline{x}(t) = \underline{x}_1(t) - \underline{x}_0(t)$




As $\Delta \underline{x}_0 \to 0$, can approximate

$$|\Delta \underline{x}(t)| \approx e^{\lambda t} |\Delta \underline{x}_0|$$

$\lambda$: Lyapunov exponent

Formally,

$$\lambda = \lim_{t \to \infty} \lim_{|\Delta \underline{x}_0| \to 0} \frac{1}{t} \ln \frac{|\Delta \underline{x}(t)|}{|\Delta \underline{x}_0|}.$$

is the <u>maximal Lyapunov exponent</u>.

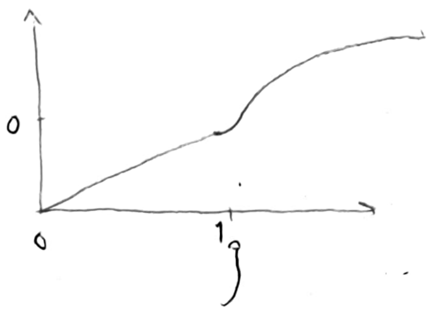
$\lambda > 0$: Sensitive dependence on initial conditions

For $g < 1$, $\lambda = g - 1$ (stable)

For $g - 1 > 0$ but small, $\lambda \approx (g-1)^2/2$

For large $g$, $\lambda \sim \ln(g)$

(Engelken, Wolf, Abbott 2023)



Suppression of chaos by stimulus (Rajan, Abbott Sompolinsky 2010):

$$\frac{dx_i}{dt} = -x_i + \sum_j J_{ij}\,\phi(x_j) + I\cos(\omega t + \underbrace{\theta_i}_{\text{random in } [0,2\pi)})$$

$g = 1.5$ 4Hz frequency

$I = 0$

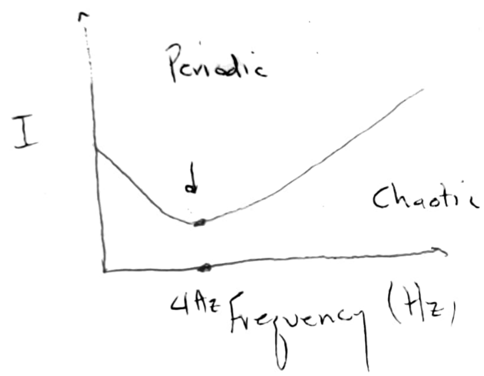

chaos

$I = 0.05$


oscillation + chaos

$I = 0.2$


oscillation




Next time: Structure in weights

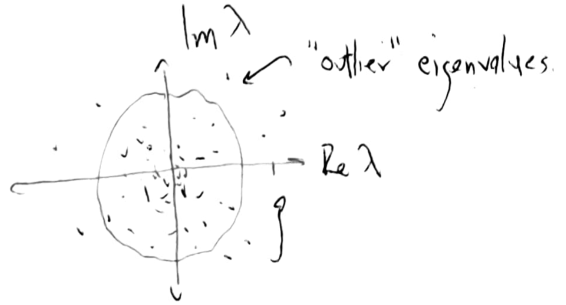
Excitatory/inhibitory eigenvalues (Rajan & Abbott 2006):

$$J = \Bigg[\ \underbrace{\sim P_E}_{f}\quad \underbrace{\sim P_I}_{1-f}\ \Bigg]$$

$P_E$ has mean $\mu_E$ and var. $g^2/N$

$P_I$ has mean $\mu_I$ and var. $g^2/N$.

If we force $\sum_j J_{ij} = 0$ exactly for all $i$, outliers are removed.




MFT & transition to chaos:

Kadmon & Sompolinsky (2015)