Rate models

$$\tau \frac{d\underline{r}}{dt} = -\underline{r} + f(W\underline{r} + \underline{h})$$

$$\tau \frac{d\underline{V}}{dt} = -\underline{V} + \underbrace{W f(\underline{V}) + \underline{h}}$$

$$C \frac{dV_i}{dt} = \sum_j g_{ij}(E_j - V_i)$$

$$\times \frac{t}{\sum_j g_{ij}} \qquad \tau(t) \frac{dV_i}{dt} = \frac{\sum_j g_{ij}(E_j - V_i)}{\sum_k g_{ik}}$$

$$= -V_i + \frac{\sum_j g_{ij} E_j}{\sum_k g_{ik}}$$

Changes in g ~ balanced

assume $\sum_j g_{ij}$ = constant

$$\underbrace{\sum_j g_{ij} E_j} \Rightarrow \underbrace{\sum_j W_{ij} r_j + h_i}$$

Integrate & fire neuron

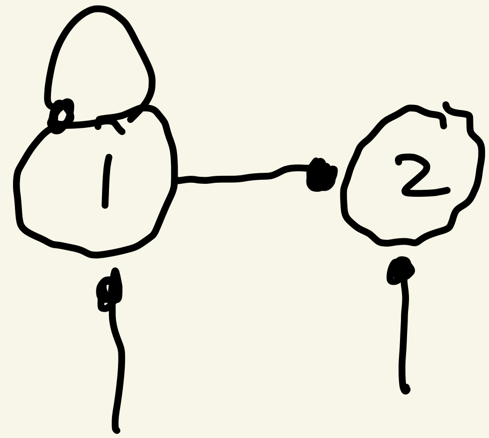
White noise ⇒ Ricciadi Eq

firing rate $f(m, \sigma)$

$$r = f(\underset{\text{wrth}}{m}, \underset{\text{variance}}{\sigma})$$

$$\tau \frac{d\underline{r}}{dt} = -\underline{r} + f(W\underline{r} + \underline{h})$$

Gain



"Hebbian" amplification

amplification $\Leftrightarrow$ slowing

"Balanced" amplification

$\neq$ slowing

Linear: $f(Wr+h) = Wr + h$

$$W\underset{\sim}{e}_i = \lambda_i \underset{\sim}{e}_i$$

neuron basis $\rightarrow$ $\underset{\sim}{r} = \sum_i r_i \underset{\sim}{e}_i$ $\leftarrow$ $\begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} = \underline{\tilde{r}}$

$$\underset{\sim}{h} = \sum_i h_i \underset{\sim}{e}_i$$

$$\underline{\tilde{r}} = E^{-1}\underline{r}$$

$$E = \begin{pmatrix} \uparrow & & \uparrow \\ \underline{e}_1 & \cdots & \underline{e}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

$$\tau \frac{d \sum_i r_i \underset{\sim}{e}_i}{dt} = -\sum_i r_i \underset{\sim}{e}_i + W \sum_i r_i \underset{\sim}{e}_i + \sum_i h_i \underset{\sim}{e}_i$$

orthog: $E^{-1} = E^{T}$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$$

$$E^{-1}E = \mathbb{1}$$

$$\underset{\sim}{l}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$$

$$\begin{pmatrix} \leftarrow \underline{l}_1^{T} \rightarrow \\ \vdots \\ \leftarrow \underline{l}_N^{T} \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & & \uparrow \\ \underline{e}_1 & \cdots & \underline{e}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

$$W\underset{\sim}{e}_i = \lambda_i e_i$$

$$WE = E\Lambda$$

$$\begin{pmatrix} \uparrow & & \uparrow \\ e_1 & \cdots & e_N \\ \downarrow & & \downarrow \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

$$E\Lambda = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 \underline{e}_1 & \lambda_2 \underline{e}_2 & \cdots & \lambda_N \underline{e}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$\underline{E^{-1}WE} = \Lambda$$

$$E^{-1}W = \Lambda E^{-1}$$

$$E^{-1} = \begin{pmatrix} \leftarrow \underline{l}_1^T \rightarrow \\ \leftarrow \underline{l}_2^T \rightarrow \\ \vdots \\ \leftarrow \underline{l}_N \rightarrow \end{pmatrix}$$

$$\underline{l}_i^T W = \lambda_i \underline{l}_i^T$$

$$\underset{\sim}{l}_i \cdot e_j = \delta_{ij}$$

$$\Lambda E^{-1} = \begin{pmatrix} \leftarrow \lambda_1 \underline{l}_1^T \rightarrow \\ \leftarrow \lambda_2 \underline{l}_2^T \rightarrow \\ \vdots \\ \leftarrow \lambda_N \underline{l}_N^T \rightarrow \end{pmatrix}$$

$$\underline{l}_j \cdot \qquad \tau \frac{d \sum_i r_i \underset{\sim}{e}_i}{dt} = -\sum_i r_i \underline{e}_i + \underbrace{W \sum_i r_i \underline{e}_i}_{\sum_i r_i (W\underline{e}_i)} + \sum_i h_i \underline{e}_i$$

$$= \sum_i r_i \lambda_i \underline{e}_i$$

$$\downarrow$$

$$l_j \cdot \qquad \tau \frac{dr_j}{dt} = -r_j + \lambda_j r_j + h_j$$

$$= \underline{-(1-\lambda_j) r_j} + h_j \qquad \lambda_j < 1$$

stable

$$\frac{\tau}{1-\lambda_j} \frac{dr_j}{dt} = -r_j + \frac{h_j}{1-\lambda_j}$$

$$\underbrace{\frac{\tau}{1-\lambda_j}}\frac{dr_j}{dt} = -r_j + \frac{h_j}{1-\lambda_j}$$

$1>\lambda_j>0$ Slowed

S.S. $r_j = \frac{h_j}{1-\lambda_j}$ $1>\lambda_j>0 \Rightarrow$ amplified

$\lambda_j<0 \Rightarrow$ diminished sped up

$\tau \to \frac{\tau}{1-\lambda_j}$

$$r_j(t) = r_j(0)\, e^{-\frac{t(1-\lambda_j)}{\tau}} + \underbrace{\frac{1}{\tau}\int_0^t dt'\, e^{-(1-\lambda_j)\frac{(t-t')}{\tau}}}\, h_j(t')$$

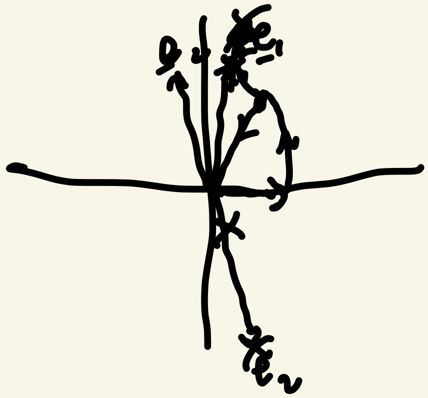
$h_j(t) \to h_j$

$h_j \frac{1}{1-\lambda_j}$ amplification

Hebbian amplification

Intuition? All $\lambda_j<1$ each componently monotonically decays

$\overset{?}{\Rightarrow}$ $\underline{r}(t)$ monotonically goes to steady state



$\underline{r}(t) = \sum r_i(t)\, \underline{e}_i$

Eigenvectors orthog

$\Rightarrow$ Normal matrix M

$\Rightarrow MM^T = M^TM$

Non-Normal

$MM^T = M^TM$

Biological W w/ E & I cells

$$\left(\begin{array}{c|c} W_{EE} & -W_{EI} \\ \hline W_{IE} & -W_{II} \end{array}\right) \begin{pmatrix} \uparrow \\ \underline{r}_E \\ \downarrow \\ \uparrow \\ r_I \\ \downarrow \end{pmatrix}$$

W_{xy}

$x \leftarrow y$

$\sum_y W_{xy} r_y$

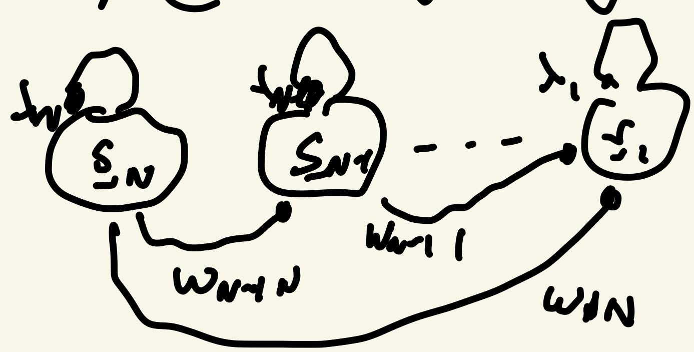
$$\left(\begin{array}{c|c} + & - \\ \hline + & - \end{array}\right) \left(\begin{array}{c|c} + & + \\ \hline - & - \end{array}\right) = \left(\begin{array}{c|c} + & + \\ \hline + & + \end{array}\right)$$

$\neq$

$$\left(\begin{array}{c|c} + & + \\ \hline - & - \end{array}\right) \left(\begin{array}{c|c} + & - \\ \hline + & - \end{array}\right) = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Schur Transformation

$$\begin{pmatrix} \lambda_1 & & W_{FF} \\ & \lambda_2 & \\ 0 & & \ddots \quad \lambda_N \end{pmatrix}$$



Schur transformation

$$\underline{e}_1 \ \underline{e}_2 \cdots \underline{e}_N$$

Gram-Schmidt Orthogonalisation

$$\underline{s}_1 = \underline{e}_1$$

$$\hat{\underline{s}}_2 = \underline{e}_2 - (\underline{s}_1 \cdot \underline{e}_2)\, \underline{s}_1$$

$$\hat{s}_3 = \underline{e}_3 \ \frac{\hat{s}_2}{\|s_2\|} - (e_3 \cdot \underline{s}_2)\, \underline{s}_2 - (e_3 \cdot s_1)\, \underline{s}_1$$

$\vdots$

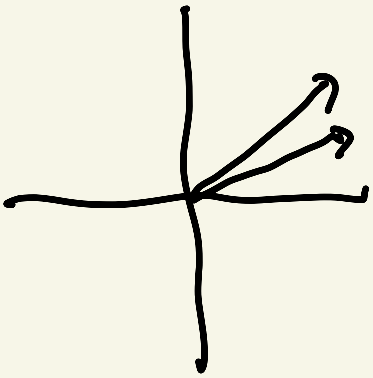
$$\begin{pmatrix} \lambda_1 & & W^{FF} \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

Orthog: preserves $\sum_{ij} \|M_{ij}\|^2$

All schur transforms of W have same $\sum_{ij} \|W^{FF}_{ij}\|^2$

$$W = \begin{pmatrix} w_E & -w_I \\ w_E & -w_I \end{pmatrix}$$

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_1 = w_E - w_I$$



$$\underline{e}_2 = \begin{pmatrix} w_I \\ w_E \end{pmatrix} = \qquad \lambda_2 = 0$$

$$\underline{s}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad W\underline{s}_1 = \lambda_1 \underline{s}_1$$

$$\underline{s}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad W$$

$$W\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} w_E + w_I \\ w_E + w_I \end{pmatrix} = (w_E + w_I)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$W\underline{s}_2 = (w_E + w_I)\,\underline{s}_1$$
$$= w_{FF}\,\underline{s}_1$$

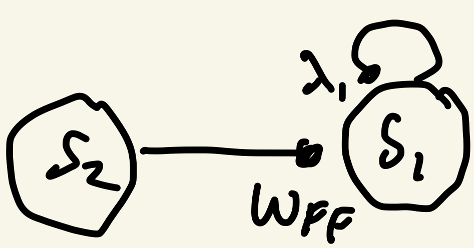
In Schur basis

$$\underbrace{W = \begin{pmatrix} \lambda_1 & w_{FF} \\ 0 & 0 \end{pmatrix}}_{W_{schur}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_2$$

w_E & w_I big

~ balanced

λ_1 small



$$W = \begin{pmatrix} \lambda_1 & W_{FF} \\ 0 & 0 \end{pmatrix} \qquad \underset{\sim}{r} = r_1 \underline{s}_1 + r_2 \underline{s}_2$$

$$\tau \frac{dr_1}{dt} = -r_1 + \lambda_1 r_1 + \underline{w_{FF} r_2 + h_s(t)}$$

$$\tau \frac{dr_2}{dt} = -r_2 + h_d(t)$$

orig basis

$$\begin{pmatrix} r_E \\ r_I \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (r_E + r_I) - \underline{s}_1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad s_1 \to \frac{1}{\sqrt{2}}$$

$$s_2 \to \frac{-1}{\sqrt{2}}$$

$$\cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (r_E - r_I) - s_2$$

$$r_2(t) = r_2(0) e^{-t/\tau} + \frac{1}{\tau} \underline{\int_0^t dt' e^{-(t-t')/\tau} h_d(t')}$$

$$r_1(t) = r_1(0) e^{-\frac{t(1-\lambda_1)}{\tau}} + \frac{1}{\tau} \int_0^t dt' \underline{e^{-\frac{(t-t')(1-\lambda_1)}{\tau}}} \left(\underline{W_{FF}} r_2(t') + h_s(t') \right)$$

$$\to \underline{\int dt' \left[e^{-\frac{(t-t')(1-\lambda_1)}{\tau}} e^{-t'/\tau} \right]} r_2(0)$$

$$\to \frac{1}{\tau} \int dt'' \left[\underline{\int dt' e^{-\frac{(t-t')(1-\lambda_1)}{\tau}} e^{-(t'-t'')/\tau}} h_d(t'') \right]$$

$$\int dt' e^{-k_1 (t-t')} e^{-k_2 (t'-t'')}$$

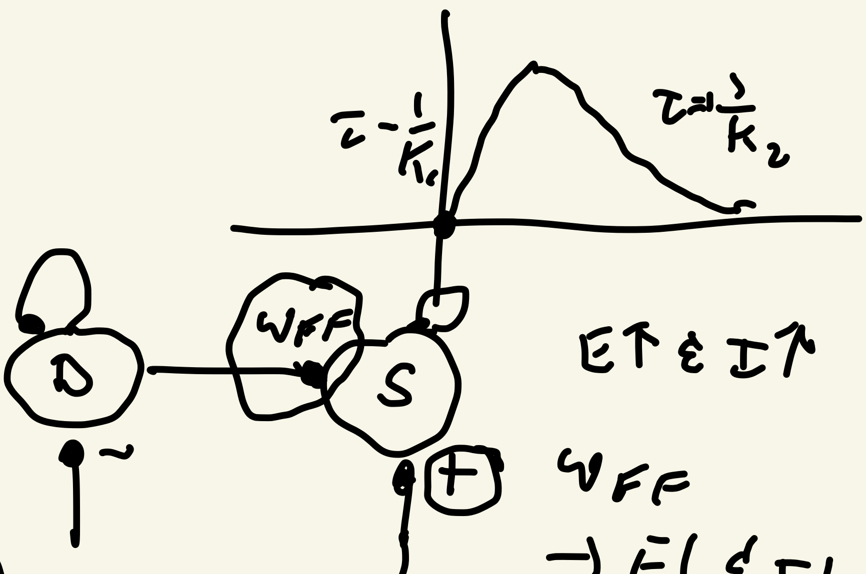
$$\int dt'\, e^{-k_1(t-t')}\, e^{-k_2(t'-t'')}$$

$$= e^{-k_1 t}\, e^{k_2 t''} \underbrace{\int_0^t dt'\, e^{-t'(k_2-k_1)}}_{\frac{1}{k_2-k_1}\left[e^{-t(k_2-k_1)} - 1\right]}$$

$$= \underbrace{\frac{1}{k_2-k_1}\left[e^{-k_2 t} - e^{-k_1 t}\right]}_{g(k_1,k_2)(t)}\, e^{k_2 t''}$$

$$\underbrace{e^{-(1-\lambda_1)\frac{t}{\tau}} - e^{-(1-\lambda_1)\frac{t}{\tau_2}}}$$

$$k_1 > k_2$$



E↓ I↑

input → I

$$\frac{1}{k_2-k_1} = \frac{\tau}{(1-\lambda_2)-(1-\lambda_1)} = \frac{\tau}{\lambda_1-\lambda_2}$$

$\boxed{w_{EE} > 1}$ ↔ $\boxed{w_{FF} \text{ sufficiently big}}$

E↓ & I↓

paradoxical