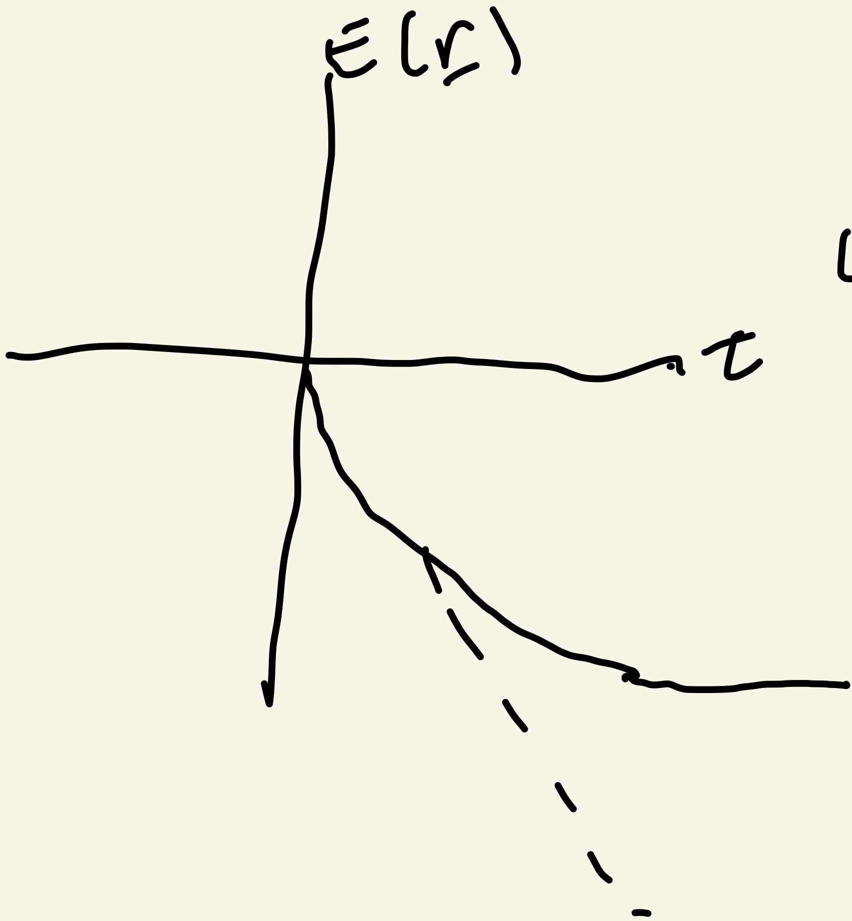
Lyapunov functions

$$\frac{d\underline{r}}{dt} = f(\underline{r})$$



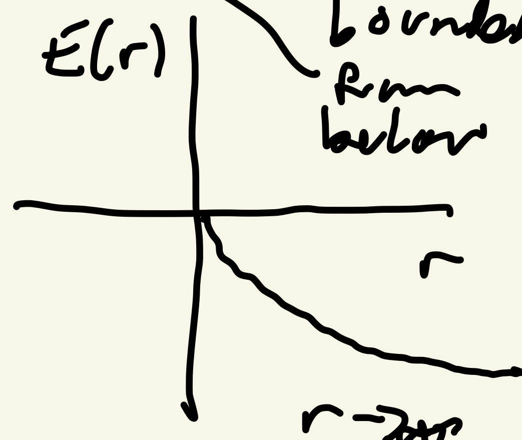
$E(\underline{r})$ $\quad \frac{dE(\underline{r})}{dt} = \nabla_{\underline{r}} E \cdot \frac{d\underline{r}}{dt}$

if (1) $\frac{dE(\underline{r})}{dt} < 0$

whenever $\frac{dr}{dt} \neq 0$

(2) boundedness conditions

$E(\underline{r})$ & $|\underline{r}|$ are bounded from below



$\Rightarrow$ (i) No periodic orbits

$r(t_1) = r(t_2)$ $\qquad \frac{dr}{dt} \neq 0$

(ii) $\Rightarrow$ stable fixed points

Stable linear system

$$\tau \frac{d\underline{r}}{dt} = -\underline{r} + W\underline{r} + \underline{h} = \underbrace{(W-\mathbb{1})}_{\lambda<0}\underline{r} + \underline{h}$$

W symmetric

$\Rightarrow$ eigenvalues real

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

$$E(\underline{r}) = -\left(\underbrace{\overbrace{\tfrac{1}{2}\underline{r}^T(W-\mathbb{1})\underline{r}}} + \overbrace{\underline{h}\cdot\underline{r}}\right)$$

$$\frac{d}{d\underline{r}} \rightarrow \underline{\underline{(W-\mathbb{1})\underline{r}}}$$

$$\tfrac{1}{2}\underline{r}^T(W-\mathbb{1}) + \tfrac{1}{2}(W-\mathbb{1})\underline{r}$$

$$\nabla_r E(\underline{r}) = -\left((W-\mathbb{1})\underline{r} + \underline{h}\right)$$

$$= -\frac{d\underline{r}}{dt}$$

$$\frac{dE(\underline{r})}{dt} = \nabla_{\underline{r}} E(\underline{r}) \cdot \frac{d\underline{r}}{dt} = -\left|\frac{d\underline{r}}{dt}\right|^2 < 0$$

$$|r| \rightarrow \infty \qquad E(r) \rightarrow -\tfrac{1}{2}\underbrace{\underline{r}^T(W-\mathbb{1})\underline{r}}_{<0}$$

$$r = \sum_i r_i \underline{e}_i$$

$$\sum_{ij} r_i \underline{e}_i^T \underline{(W-\mathbb{1})} r_j \underline{e}_j$$

$$\rightarrow \lambda_j \underline{e}_j$$

$|r| \rightarrow \infty$

$E(r) \rightarrow +\infty$

$\Rightarrow$ <u>fixed point</u>

$$= \sum_{ij} r_i r_j \lambda_j \underbrace{\underline{e}_i^T \underline{e}_j}_{\delta_{ij}}$$

$$= \sum_i r_i^2 \lambda_i < 0$$

Hopfield, Grossberg

$$\frac{d\underline{v}}{dt} = -\underline{v} + \underbrace{W f(\underline{v})}_{\underline{r}} + \underline{h}$$

<u>f bounded</u>

$f' > 0$

<u>W symmetric</u>

f bounded $\Rightarrow |\underline{v}|$ bounded

$$E = -\underbrace{\sum_i \int_0^{r_i} f^{-1}(x)\,dx} + \frac{1}{2}\underline{r}^T W \underline{r} + \underline{h}\cdot\underline{r}$$

$$(\nabla_{\underline{r}} E)_i = \frac{dE}{dr_i} = \sum_i \underset{-\underline{v}}{- f^{-1}(r_i)} + W\underline{r} + \underline{h} = \frac{dv}{dt}$$

$f(v) = r$

$f^{-1}(r) = v$

$$\frac{dE}{dt} = \nabla_r E \cdot \frac{dr}{dt} = \underbrace{\frac{dr}{dt}}_{\underset{>0}{f'}\frac{dv}{dt}} \frac{dv}{dt}$$

$$= f'\left(\frac{dv}{dt}\right)^2 > 0$$

or $= 0$ if $\frac{dv}{dt} = 0$

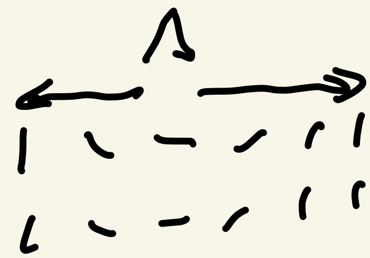
Developmental Models

(1) Feature-based   low-D

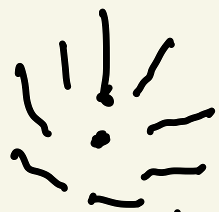
$$z = re^{i2\theta}$$

θ = pref ori
r = ori selectivity

$$\frac{dz}{dt} = F(z)$$



Wolf & colleagues



pinwheel

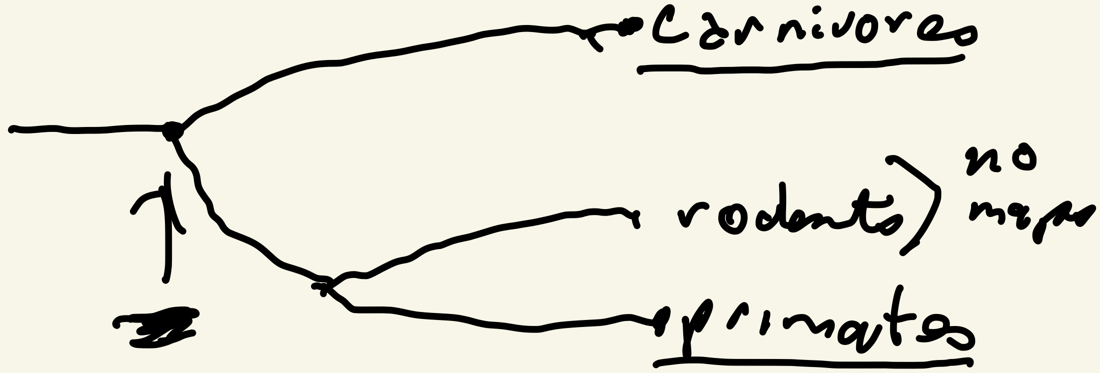
suppressive long-range interaction → $> \Lambda$

→ π pinwheels / Λ^2

100 maps ~ 10K pinwheels

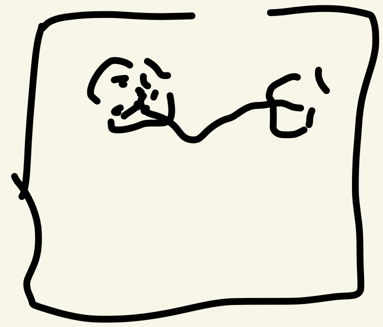
$\boxed{\pi \pm 2\%}$

Galago, Tree shrew (primates), ferret (carnivore)



High-D

- Hebbian rule competitive
- Activity clusters



One post cell

y post activity

$\underline{x}$ pre activity

$w(x_i) = w_i$

$\underline{w}$

Hebb: $\Delta \underline{w} = f_y(y)\, f_{\underline{x}}(\underline{x}) \rightarrow (f_x(x_1), \dots f(x_N))^T$

Activity $y = g(\underline{w} \cdot \underline{x})$

$$f_y(g(w \cdot x)) \rightarrow f_y(\underline{w} \cdot \underline{x})$$

$$\Delta \underline{w} = f_y(\underline{w} \cdot \underline{x})\, f_{\underline{x}}(\underline{x})$$

$\tau_w \frac{d\underline{w}}{dt} = \dots$

slow

$$\langle \Delta \underline{w} \rangle_t = \langle f_y(\underline{w} \cdot \underline{x})\, f_x(\underline{x}) \rangle_t$$

$$\Delta \underline{w} = \langle f_y(\underline{w} \cdot \underline{x})\, f_x(\underline{x}) \rangle_{\underline{x}}$$

$$f_y(\underline{w} \cdot \underline{x}) = \underline{f}_y(\underline{x}) \cdot \underline{w}$$

$$\underbrace{\langle f_y(\underline{x})\, f_{\underline{x}}(\underline{x}) \rangle}_{Q_{ij} = Q_{ji}} w$$

$Q_{ij} = \langle f_y(x_i)\, f_x(x_j) \rangle$

Covariance rule

$$Q = \left\langle (\underline{x} - \langle \underline{x} \rangle)(\underline{x} - \langle \underline{x} \rangle)^T \right\rangle_{\underline{x}}$$

$$\boxed{\Delta \underline{w} = Q \underline{w}}$$

cov: $\lambda_Q \geq 0$

$$\underline{e}_i^T Q \underline{e}_i = \lambda_i$$

$$e_i^T \langle (x - \langle x \rangle)(x - \langle x \rangle)^T \rangle \underline{e}_i$$

$$\langle (\underbrace{(x - \langle x \rangle)^T e_i}) ^T \underbrace{(x - \langle x \rangle)^T \underline{e}_i}_{\geq 0} \rangle$$

Behavior

Leading eigenvectors (biggest λ) will dominate development

w_i = constant unstable fixed pt

linearize $\Rightarrow$ leading eig's of linear dynamics will dominate

+ constrain

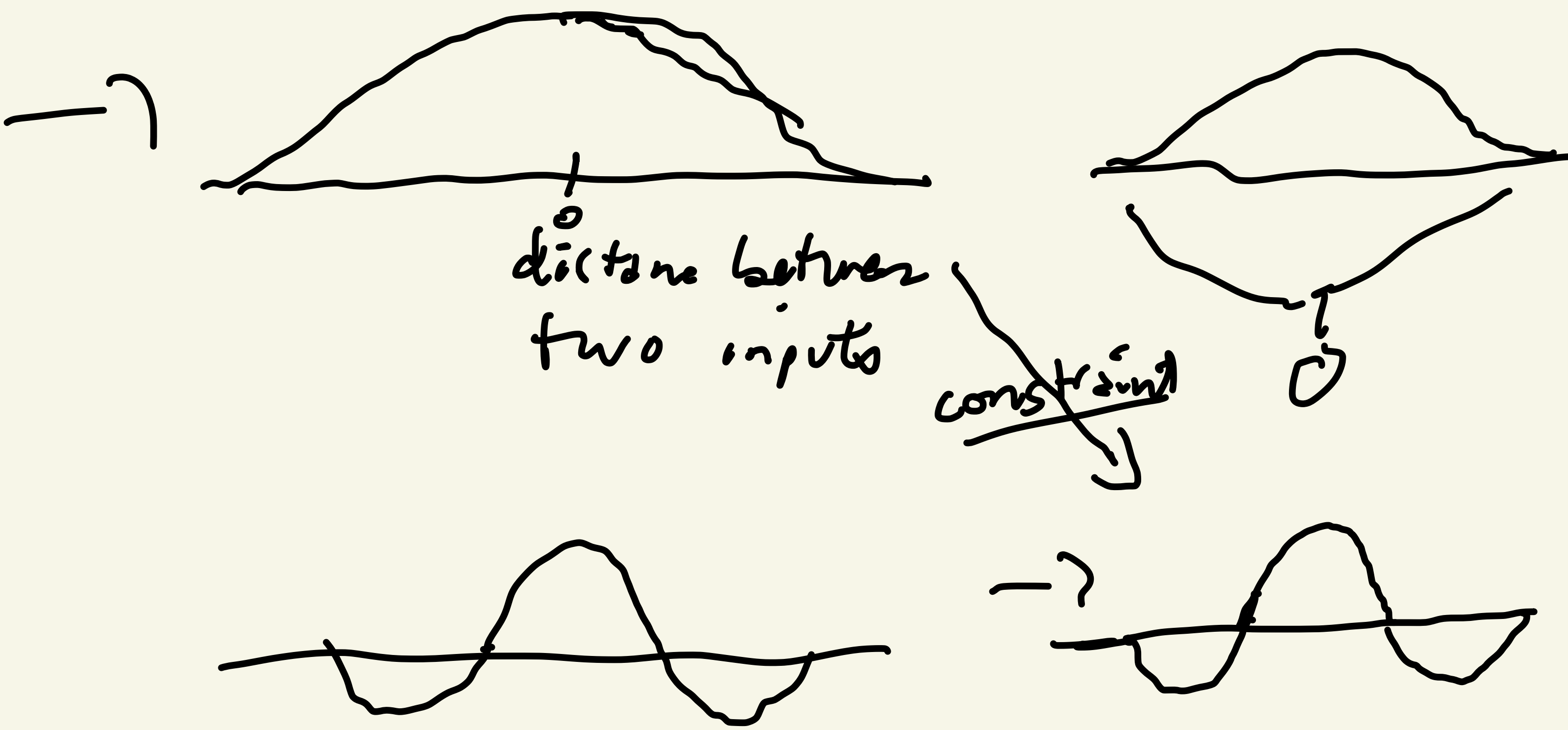
→ make competitive (be selective)

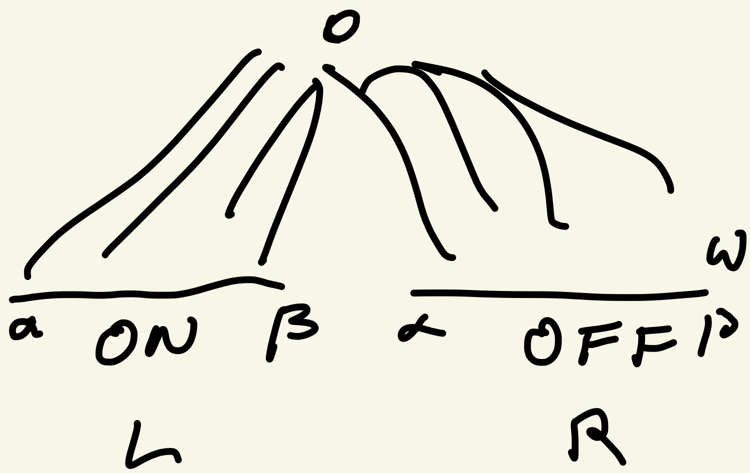
max, min synaptic weight

$\underline{w_i \geq 0}$ $\underline{w_i \leq w_{max}}$

What do principal eigenvectors look like?

Correlation (Q) princ eig





$w^L \geq 0$

$w^R \geq 0$

$\underline{w^D(\alpha) = w^L(\alpha) - w^R(\alpha)}$

Constrain $\Rightarrow$ competitive

$\boxed{\sum_i w_i = c}$ $\boxed{\langle y \rangle_x \sim c}$ homeostatic

$\sum_i w_i^2 = c$

$\sum_i w_i \to \underline{w}\cdot\underline{n}$ $\quad n = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$

$\Delta \underline{w} = Q\underline{w}$ + subtract $\frac{(\sum_i w_i - c)}{N}$ from all i

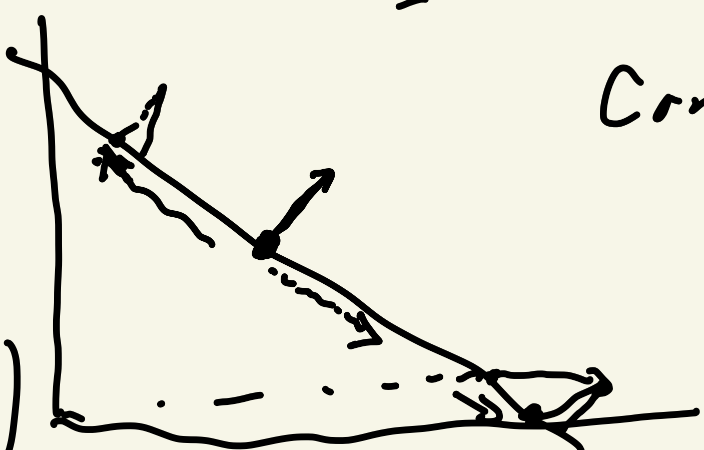
$$\boxed{\Delta w = Q\underline{w} - \frac{\underline{n}^T Q \underline{w}}{\underline{n}^T\cdot\underline{n}}\,\underline{n}}$$

$\Rightarrow \sum_i \Delta w_i = 0$

$n\cdot\Delta w = 0$

$$n\cdot\Delta w = n^T Q\underline{w} - \frac{n^T Q\underline{w}}{\cancel{\underline{n}\cdot\underline{n}}}\cancel{\underline{n}\cdot\underline{n}} = 0$$

Constrain $\underline{n}$ direction



$\Delta W = \underline{P}QW$

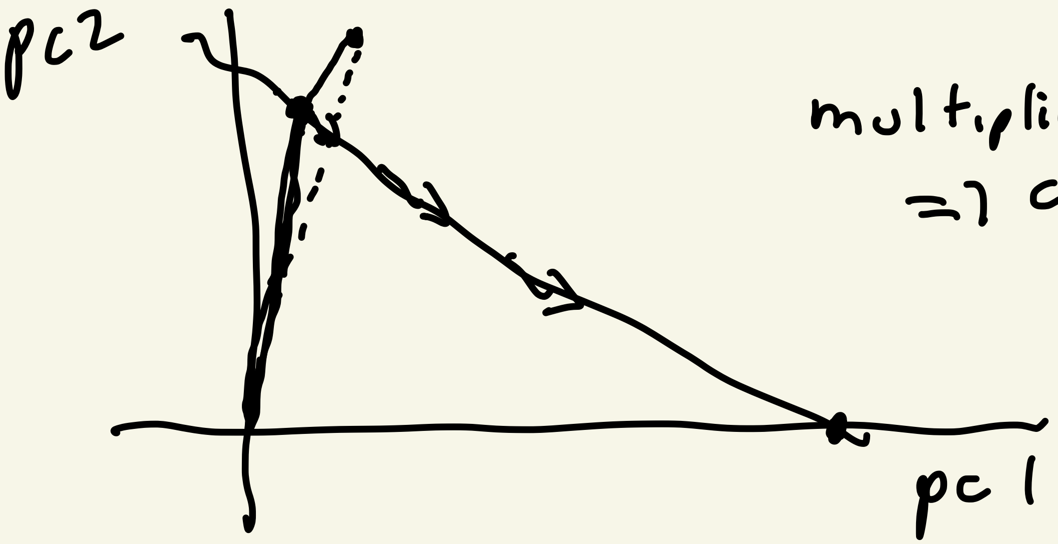
$P = \left(\mathbb{1} - \frac{\underline{n}\,\underline{n}^T}{\underline{n}^T\underline{n}}\right)$

$\Delta\underline{w} = Q\underline{w}$ + multiply to set $\sum_i w_i = c$

$= Qw - (\quad)\,\underline{w}$

$- \frac{n^T Q w}{n^T\underline{w}}\,\underline{w}$

$$\underline{n}\cdot\Delta\underline{w} = \underline{n}^T Q\underline{w} - \frac{\underline{n}^T Q w}{\cancel{n^T w}}\cancel{n^T w} = 0$$



multiplicatively

$\Rightarrow$ converge to pc1

$$\underline{w} = w_1 \underline{e}_1 + w_2 \underline{e}_2$$

$$Qw = \lambda_1 w_1 \underline{e}_1 + \lambda_2 w_2 \underline{e}_2$$

$$\lambda_1 > \lambda_2$$

$$\frac{w_1}{w_2} < \frac{\lambda_1 w_1}{\lambda_2 w_2}$$

Subtractive:

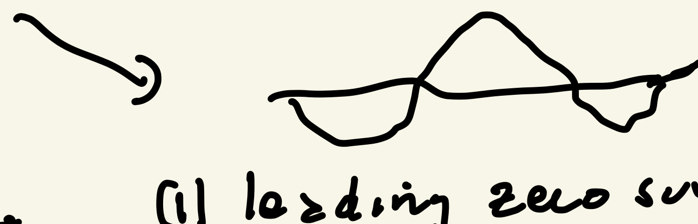
$\underline{e}_i$: eigenvectors:

if $\underline{e}_i \cdot \underline{n} = 0$ unchanged grows at λ_i

if $\underline{e}_i \cdot \underline{n} \neq 0$ $\underline{e}_i \rightarrow$ constrained mode $\tilde{\underline{e}}_i \perp \underline{n}$

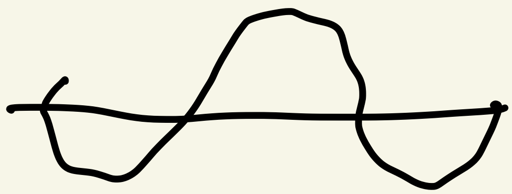
smaller $\tilde{\lambda}_i < \lambda_i$

pc1



pc2

$\cdot\, \underline{n} = 0$



(1) leading zero sum eig

or (2) leading constrained eigv

Oja's rule

$$\Delta \underline{w} = y(\underline{x} - \underline{w}\, y)$$

multiplicatively constraint on $|w| = 1$

$y = \underline{w} \cdot \underline{x}$

$\langle y \underline{x} \rangle = \langle x x^T \rangle w = Q w$

$-\langle y^2 \rangle w$ multiplicative

$y = \underline{w} \cdot \underline{x}$

$\langle y^2 \rangle = w^T \langle x x^T \rangle w = w^T Q w$

$$\boxed{Q \underline{w} - (\underline{w}^T Q \underline{w})\, \underline{w}}$$

$\underline{w} \cdot \underline{w} = 1$

$w \cdot \Delta w < 0$ if $w \cdot w \geq 1$

$w \cdot \Delta w \geq 0$ if $w \cdot w < 1$

$\underline{w} \cdot$

$= 0$

$$Q \underline{w} - \frac{w^T Q \underline{w}}{\underline{w} \cdot \underline{w}}\, \underline{w}$$

$\underline{w} \cdot \Delta \underline{w} = 0$ conserves $|w|$

$\Delta |w|^2 = 0$

Apply to OD

L & R

constraint $\underline{w^L + w^R}$

$\underline{w^L - w^R}$ unconstrained

ON & OFF

$w^N - w^F$ unconstrained

$w^D = w^L - w^R$ or $w^N - w^F$

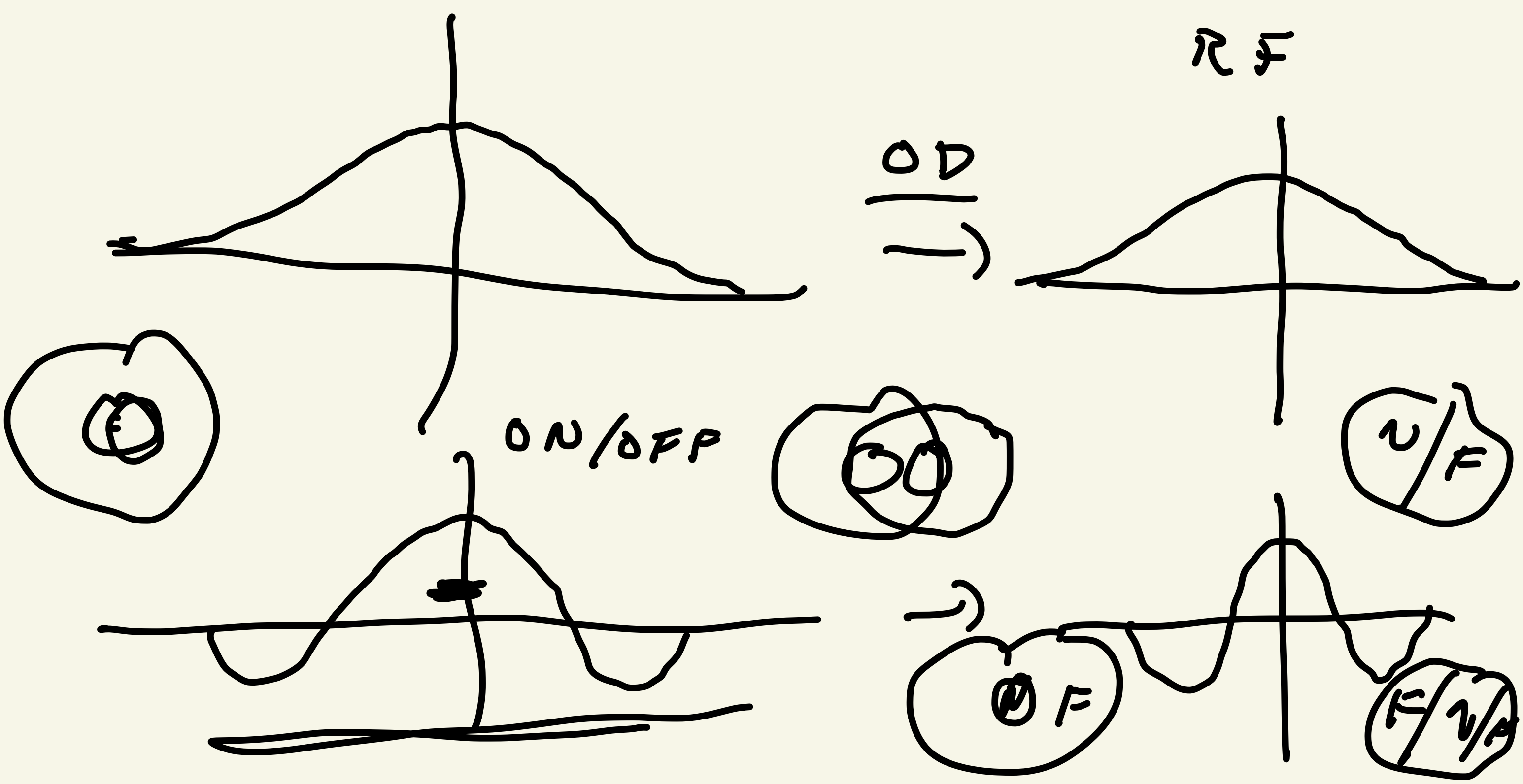
driven

$C^D = C^{SAME} - C^{OPP}$

$\Delta w^D = C^D w^D$

CD

RF



Ohshiro & Weliky 2006