Assignment 9: Addendum

In the last (optional) problem of simulating the dynamics: I referred to initializing weights in the vicinity of 1, and setting a lower weight limit at zero. An alternative approach you might take is to imagine you are simulating the difference between two input populations, $\mathbf{v}^1$ and $\mathbf{v}^2$. These might represent inputs from the left and right eyes, where the difference represents ocular dominance; or inputs from ON- and OFF-center inputs, where the difference represents the alternation between ON subregions and OFF subregions in simple-cell receptive fields. (You can also consider all 4 input types, if you're interested in this see a paper by E. Erwin & K.D. Miller, J. Neurosci., 1998.) We assume a symmetry between the two types, so that correlations depend only on whether two inputs are of the same type, specified by a correlation matrix $\mathbf{C}^S$ ('s' for 'same'), or or are of opposite types, specified by a correlation matrix $\mathbf{C}^O$ ('o' for 'opposite'); they do not depend specifically on which input is of a given type. Suppose we have a constraint that conserves $\mathbf{p} \cdot \mathbf{v}$ ($i.e.$, $\mathbf{p} = \mathbf{A}$): if implemented subtractively it gives a term $-\lambda^S(\mathbf{v})\hat{\mathbf{p}}$ (if the overall unconstrained equation is $\frac{d}{dt}\mathbf{v} = \mathbf{L}\mathbf{v}$, where $\mathbf{v} = \begin{pmatrix} \mathbf{v}^1 \\ \mathbf{v}^2 \end{pmatrix}$, then $\lambda^S(\mathbf{v}) = \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{L}\mathbf{v}$), or a multiplicative constraint, which we will write as $-\lambda^M(\mathbf{v})\hat{\mathbf{v}}$ ($\lambda^M(\mathbf{v}) = \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{L}\mathbf{v}/(\hat{\mathbf{p}} \cdot \hat{\mathbf{v}})$).

Then if we assume linear equations with constraints, let $\mathbf{A}$ now represent the diagonal matrix whose diagonal entries are $A(\mathbf{r})$, and include both types of constraints to see what happens to each of them, we have

$$\frac{d}{dt}\mathbf{v}^1 = \mathbf{C}^S\mathbf{A}\mathbf{v}^1 + \mathbf{C}^O\mathbf{A}\mathbf{v}^2 - \lambda^S(\mathbf{v})\hat{\mathbf{p}} - \lambda^M(\mathbf{v})\hat{\mathbf{v}^1} \tag{1}$$

$$\frac{d}{dt}\mathbf{v}^2 = \mathbf{C}^O\mathbf{A}\mathbf{v}^1 + \mathbf{C}^S\mathbf{A}\mathbf{v}^2 - \lambda^S(\mathbf{v})\hat{\mathbf{p}} - \lambda^M(\mathbf{v})\hat{\mathbf{v}^2} \tag{2}$$

We now transform to sum and difference coordinates: $\mathbf{v}^{Sum} = \mathbf{v}^1 + \mathbf{v}^2$, $\mathbf{v}^D = \mathbf{v}^1 - \mathbf{v}^2$ (you could multiply both right sides by $1/\sqrt{2}$ but I'll neglect that). We let $\mathbf{C}^{Sum} = \mathbf{C}^S + \mathbf{C}^O$ and $\mathbf{C}^D = \mathbf{C}^S - \mathbf{C}^O$. By adding or subtracting Eqs. 1 and 2, we obtain the corresponding equations:

$$\frac{d}{dt}\mathbf{v}^{Sum} = \mathbf{C}^{Sum}\mathbf{A}\mathbf{v}^{Sum} - 2\lambda^S(\mathbf{v})\hat{\mathbf{p}} - \lambda^M(\mathbf{v})\hat{\mathbf{v}} \tag{3}$$

$$\frac{d}{dt}\mathbf{v}^D = \mathbf{C}^D\mathbf{A}\mathbf{v}^D - \lambda^M(\mathbf{v})\hat{\mathbf{v}^D} \tag{4}$$

(If we assume nonlinear equations for Eqs. 1 and 2, then, if we assume that the initial condition for $\mathbf{v}^D$ is small fluctuations around $\mathbf{v}^D \equiv 0$, we can linearize the equations for $\mathbf{v}^D$ about the fixed point $\mathbf{v}^D \equiv 0$ and thus arrive at a constrained linear equation for the initial development of $\mathbf{v}^D$ like that above).

We thus see that, if we are simulating the development of $\mathbf{v}^D$, then subtractive constraints do not affect this development, only multiplicative constraints will apply to $\mathbf{v}^D$; so you can either simulate unconstrained (if you imagine subtractive constraints were used) or else multiplicatively constrained. If $\mathbf{v}^1$ and $\mathbf{v}^2$ have minimum and maximum weights of, say, 0 and 4, then $\mathbf{v}^D$ will have minimum and maximum weights of -4 and 4. The initial condition of $\mathbf{v}^D$ will be random noise about $\mathbf{v}^D = 0$ instead of about $\mathbf{v}^D = 1$. With these changes, you can consider that you are simulating $\mathbf{v}^D$ rather than a single population with non-negative weights.