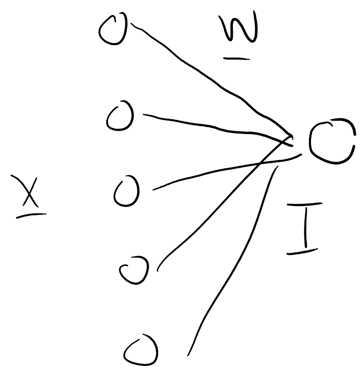
Vector $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \in \mathbb{R}^N$



"perceptron"

$I = \underline{w} \cdot \underline{x}$

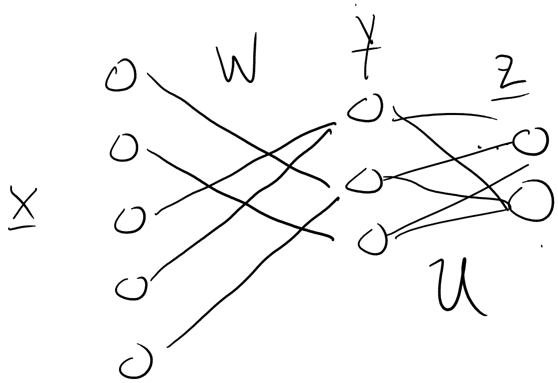
$r = \begin{cases} 1 & I > \theta \\ 0 & I \leq \theta \end{cases}$

Matrix $A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{M1} & & & A_{MN} \end{pmatrix} \in \mathbb{R}^{M \times N}$

Matrix multiplication $\underline{y} = W \underline{x}$

$y_j = \sum W_{jk} x_k$

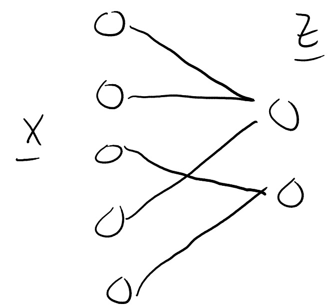
Ex:



$$\underline{z} = U\underline{y} = UW\underline{x}$$

$$z_i = \sum_j U_{ij} \sum_k W_{jk} x_k$$

$$= \sum_k \overbrace{\boxed{\sum_j U_{ij} W_{jk}}}^{W^{eff}} x_k$$



$$W^{eff} = UW$$

Not true if nonlinear: $\underline{z} = f(U\underline{y}),\ \underline{y} = f(W\underline{x})$

$$= f(Uf(W\underline{x})) \neq f(UW\underline{x})$$

Recall:

Linear independence, rank, null spaces, spans, bases.

Eigendecomposition:

$A \in \mathbb{R}^{N \times N}$, $A\underline{v}$ maps $\underline{v}$ to a new vector in $\mathbb{R}^N$

Square matrix $A \in \mathbb{R}^N$ has eigenvalue $\lambda \in \mathbb{C}$ if

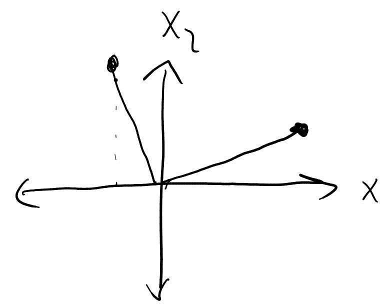
$A\underline{v} = \lambda \underline{v}$; $\underline{v} \in \mathbb{C}^N$ is associated eigenvector.

__Ex 1:__ $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\quad A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_1 = 5$

$A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\quad \lambda_2 = 1$

__Ex 2:__ $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

λ_1

$\underline{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad A\underline{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ $\quad \lambda_2 = i$

Complex eigenvalues $\rightarrow$ rotation. Come in conjugate pairs $\lambda = \alpha \pm \beta i$

If e-vals. all distinct, e-vecs. form basis of $\mathbb{R}^N$.

(defective e-vals $\longrightarrow$ generalized e-vecs).

Can represent any $\underline{x} = \sum_{i=1}^{N} w_i \underline{v}_i$

$$A\underline{x} = \sum_{i=1}^{N} w_i A\underline{v}_i = \sum_{i=1}^{N} w_i (\lambda \underline{v}_i)$$

$\underline{w}$ can be thought of as eigenvector representation of $\underline{x}$ (change of basis)

$$\underline{x} = V\underline{w}, \quad V = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

$$A\underline{x} = AV\underline{w} = V\underline{\Lambda}\underline{w} \qquad \underline{\Lambda} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

Holds for all $\underline{w}$: $AV = V\underline{\Lambda}$,

$A = V\underline{\Lambda}V^{-1}$ "diagonalization"

Linear stability:

Rate network $\frac{d\underline{r}(t)}{dt} = \dot{\underline{r}}(t) = -\underline{r} + W\underline{r} = \underbrace{(-I+W)}_{A}\underline{r}$

If A has e-vals. $\lambda_1 \dots \lambda_N$ and e-vecs. $\underline{v}_1 \dots \underline{v}_N$, then

$$\underline{r}(t) = \sum_{i=1}^{N} w_i(t)\, \underline{v}_i$$

$$\dot{\underline{r}}(t) = \sum_{i=1}^{N} \dot{w}_i(t)\, \underline{v}_i = A\underline{r}(t) = \sum_{i=1}^{N} w_i(t)\, \lambda_i \underline{v}_i$$

Satisfied if, $\forall i$, $\dot{w}_i(t) = \lambda_i w_i(t)$

$\underline{v}_i$: mode of pop. activity

$w_i(t)$: time-dependent strength of i^{th} mode

$$w_i(t) = w_i(0)\, e^{\lambda_i t}$$

$$\lambda = \alpha + \omega i, \quad \alpha = Re[\lambda], \quad \omega = Im[\lambda]$$

$$e^{\lambda t} = e^{(\alpha + \omega i)t} = e^{\alpha t} e^{\omega i t} = \underbrace{e^{\alpha}}_{\text{exp. growth/decay}} \underbrace{(\cos \omega t + i \sin \omega t)}_{\text{oscillations}}$$

$\alpha < 0$: exp. decay

$\alpha > 0$: exp growth

$\alpha = 0$: neither

$\omega \neq 0$: oscillations

As $t \to \infty$, mode with largest $Re[\lambda_i]$ dominates

Classification of fixed pts. $\quad \dot{\underline{r}} = A\underline{r} + \underline{I}$

$$= A(\underline{r} + \underline{b}), \quad \underline{b} = A^{-1}\underline{I}$$

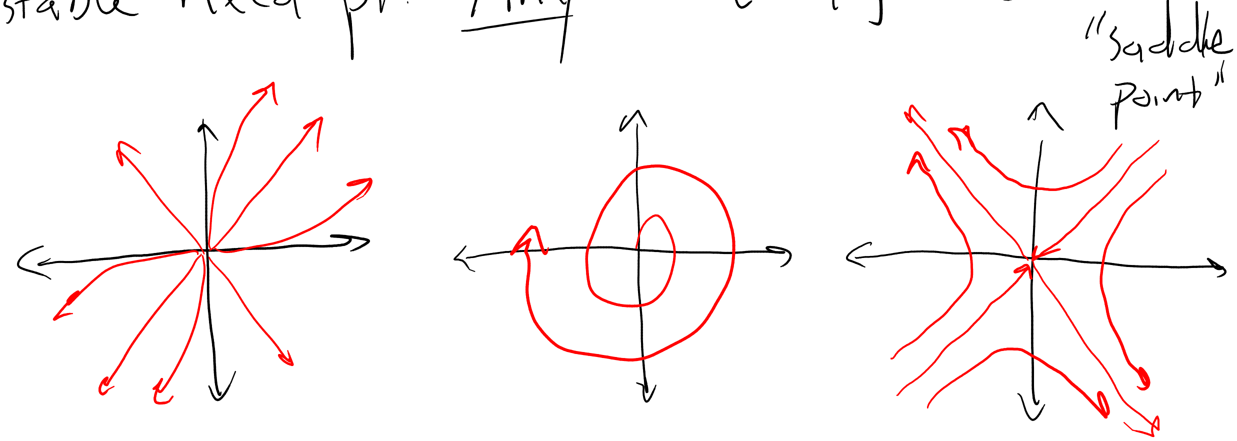
Let $\underline{r} + \underline{b} \Rightarrow \underline{r}$, then $\qquad$ Fixed pt: $\underline{r} = -\underline{b}$

$\dot{\underline{r}} = A\underline{r}$, fixed pt. at $\underline{0}$.

Stable fixed pt: All $\mathrm{Re}[\lambda_i] < 0$.

As $t \to \infty$, $\underline{r} \to \underline{0}$

Unstable fixed pt: $\underline{\text{Any}}$ $\mathrm{Re}[\lambda_i] > 0$



Linearization of nonlinear systems:

$$\underline{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} \qquad \dot{r}_i = f_i(r_1 \ldots r_N)$$

<u>Partial derivative</u>

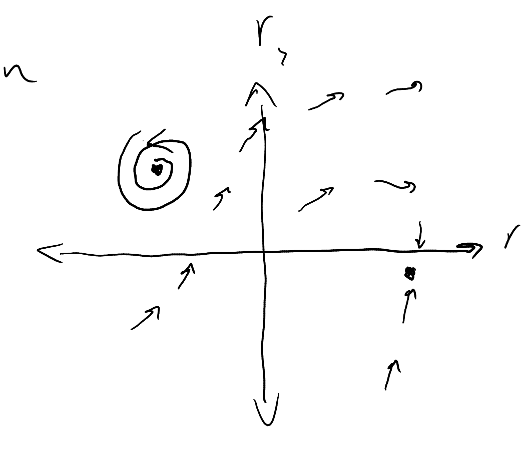
$$\left.\frac{\partial f_i}{\partial r_j}\right|_{\underline{r}} = \lim_{\Delta r_j \to 0} \frac{f_i(r_1 \ldots r_j + \Delta r_j \ldots) - f_i(r_1 \ldots r_N)}{\Delta r_j}$$

(other vars. held fixed)

Jacobian:
$$J(\underline{r}) = \left.\begin{pmatrix} \partial f_1/\partial r_1 & \partial f_1/\partial r_2 & \cdots & \partial f_N/\partial r_N \\ \partial f_2/\partial r_1 & \partial f_2/\partial r_2 & & \vdots \\ \vdots & & \ddots & \\ \partial f_N/\partial r_1 & \cdots & & \partial f_N/\partial r_N \end{pmatrix}\right|_{\underline{r}}$$

$\delta \underline{r} = \underline{r}(t) - \underline{r}_0$ small, then

$$\delta \dot{\underline{r}}(t) \simeq J(\underline{r}_0)\, \delta \underline{r}(t)$$



Suppose $f(\underline{r}_0) = 0$ (fixed pt.),

$\{\lambda_i\} = \text{eig}(J(\underline{r}_0))$

If $\text{Re}[\lambda_i] < 0 \ \forall i$, $\underline{r}_0$ is stable.

If $\text{Re}[\lambda_i] > 0$ for any i, unstable

If $\text{Re}[\lambda_i] \leq 0$, cannot conclude stability.