April 14, 2020

I just wanted to briefly correct the point I got hung on re Lyapunov functions in today's lecture. You'll recall we had the equation

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{W}\mathbf{f}(\mathbf{v}) + \mathbf{h} \tag{1}$$

with $\mathbf{W}$ symmetric. We asserted the Lyapunov function

$$E = -\sum_i \int_0^{r_i} f^{-1}(x)dx + \frac{1}{2}\mathbf{r}^{\mathrm{T}}\mathbf{W}\mathbf{r} + \mathbf{h}^{\mathrm{T}}\mathbf{r} \tag{2}$$

In class, I got confused by the fact that there were vectors in the last two terms for E but only individual elements in the first term. The last two terms are scalars, unchanged by any coordinate transformation to orthonormal basis vectors (*i.e.*, by an orthogonal transformation); but the first, which depends on individual elements of $\mathbf{r}$, depends on the coordinate system. But this is easily fixed; if the orthonormal basis vectors of the current coordinate system are $\mathbf{e}_i$, so that $\mathbf{r} = \sum_i r_i \mathbf{e}_i$, then we can replace the first term with $\sum_i \int_0^{\mathbf{r}^{\mathrm{T}}\mathbf{e}_i} f^{-1}(x)dx$, which is invariant under orthogonal coordinate transformations. And then we can choose to do the calculation in the current coordinate system, in which case the equation for E becomes Eq. 2.